The propagation of small disturbances in a radiating gas

By WILBERT J. LICK
Harvard University, Cambridge, Massachusetts

(Received 2 July 1963)

The influence of radiation on the propagation of small disturbances is investigated by considering a signalling problem and obtaining approximate limiting solutions. It is found that, for small time, the main disturbance travels at the isentropic speed of sound. For some intermediate time, the main disturbance travels at the isothermal speed of sound. For long time, the disturbance travels at the isentropic speed of sound. The decay and diffusion of these waves are also determined.

1. Introduction

The problem of the propagation of small disturbances in a radiating gas has been previously examined by Vincenti & Baldwin (1962). These authors obtained an integro-differential equation for the velocity potential ϕ. They then studied solutions that were periodic in time. Although some physical insight may be obtained from these periodic solutions, it is more revealing to consider a general signalling problem due to a disturbance starting at $t = 0$ with conditions given on $x = 0$. In this situation, periodic waves with all possible frequencies are present and combine to form travelling waves which may decay and diffuse with time.

In order to solve this signalling problem readily, a differential equation for ϕ is obtained from the basic integro-differential equation by a substitute-kernel technique previously shown to be quite accurate (Krook 1955; Lick 1963). The differential equation obtained by this method is

$$\frac{\partial^3}{\partial t\,\partial x^2}\left(\frac{\partial^2\phi}{\partial t^2}-a_S^2\frac{\partial^2\phi}{\partial x^2}\right)+a_1^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\phi}{\partial t^2}-a_T^2\frac{\partial^2\phi}{\partial x^2}\right)-b_1^2\frac{\partial}{\partial t}\left(\frac{\partial^2\phi}{\partial t^2}-a_S^2\frac{\partial^2\phi}{\partial x^2}\right)=0, \quad (1.1)$$

where a_1^2 and b_1^2 are constants, and a_S and a_T are respectively the isentropic and isothermal sound speeds in the undisturbed medium.

The above equation is similar to the wave equation studied extensively by Whitham (1959), which is

$$\left(\frac{\partial}{\partial t}+c_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+c_2\frac{\partial}{\partial x}\right)\phi+\lambda\left(\frac{\partial}{\partial t}+a\frac{\partial}{\partial x}\right)\phi=0, \quad (1.2)$$

where c_1, c_2 and a are different wave speeds and λ is a known constant. For equation (1.2), Whitham has shown that for small time the higher-order term dominates, while the lower-order term will ultimately describe the main disturbance. In addition, the lower-order term will produce an exponential damping of the wave described by the higher-order term. In turn, the higher-order term will produce a diffusion of the wave described by the lower-order term.

By extending these results to equation (1.1), it is natural to conjecture that for small time the first term dominates. The initial wave travels at the isentropic speed of sound and is exponentially damped due to the second term. For very long time, the third term dominates. Therefore, the wave again travels at the isentropic speed, but is diffused due to the second- and higher-order terms. For some intermediate time, the second term dominates. The wave travels at the isothermal speed of sound, is diffused due to the first term and exponentially damped because of the presence of the third term. The possibility of this type of motion is examined in the present paper.

2. Basic equations

General formulation of the problem

The particular problem considered is that of the propagation of small disturbances through a semi-infinite radiating gas bounded by an infinite, plane, radiating wall at $x = 0$. Effects of viscosity and heat conductivity are neglected. The energy density and pressure of radiation are assumed to be small by comparison with the energy density and pressure of the gas and are neglected. For a perfect gas with constant specific heats, the equations of state and the equations of conservation of mass, momentum, and energy are then

$$p^* = \rho^* R T^*, \tag{2.1}$$

$$\frac{\partial \rho^*}{\partial t} + u^* \frac{\partial \rho^*}{\partial x} + \rho^* \frac{\partial u^*}{\partial x} = 0, \tag{2.2}$$

$$\frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial x} + \frac{1}{\rho^*}\frac{\partial p^*}{\partial x} = 0, \tag{2.3}$$

$$\rho^* c_P \frac{\partial T^*}{\partial t} + \rho^* u^* c_P \frac{\partial T^*}{\partial x} - \frac{\partial p^*}{\partial t} - u^* \frac{\partial p^*}{\partial x} = Q_r, \tag{2.4}$$

where p^*, ρ^*, T^*, u^* are respectively the pressure, density, temperature and velocity of the gas at the position x and time t. R is the gas constant and c_P is the specific heat at constant pressure. Q_r is the net radiative heat input to the gas per unit volume per unit time. In terms of q_r, the net radiative heat flux, Q_r is given by

$$Q_r = -\partial q_r/\partial x. \tag{2.5}$$

If it is assumed that the radiative absorption coefficient α is independent of frequency and temperature, q_r is given by (Kourganoff 1952; Lick 1963)

$$q_r = 2\alpha\sigma \int_0^x T^{*4} E_2[\alpha(x-\bar{x})]\, d\bar{x} - 2\alpha\sigma \int_x^\infty T^{*4} E_2[\alpha(\bar{x}-x)]\, d\bar{x} + 2\epsilon_W \sigma T_W^4 E_3(\alpha x). \tag{2.6}$$

E_n is the exponential integral defined as

$$E_n(t) = \int_0^1 \mu^{n-2} e^{-t/\mu}\, d\mu, \tag{2.7}$$

σ is the Stefan–Boltzmann constant and ϵ_W is the emissivity of the wall, hereafter assumed to be one.

For time $t < 0$, it is assumed that the gas is at rest with temperature T_0, pressure p_0, and density ρ_0. The wall is stationary with temperature T_0. For $t \geqslant 0$, the wall is moved with a constant velocity β and simultaneously its temperature is raised to $T_0 + \delta/R$ where δ is a constant.

Linearization and kernel substitution

Equations (2.1)–(2.7) may be simplified by the usual process of linearization. If we confine our attention to phenomena such that

$$\rho^* = \rho_0 + \rho, \quad p^* = p_0 + p, \quad T^* = T_0 + T, \quad u^* = u,$$

where ρ, p, T and u are small perturbations, then equations (2.2)–(2.4) become

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0, \tag{2.8}$$

$$\frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0, \tag{2.9}$$

$$\rho_0 c_P \frac{\partial T}{\partial t} - \frac{\partial p}{\partial t} = Q_r. \tag{2.10}$$

By differentiating the equation of state and linearizing, one obtains

$$R\frac{\partial T}{\partial t} = \frac{1}{\rho_0}\frac{\partial p}{\partial t} - \frac{p_0}{\rho_0^2}\frac{\partial \rho}{\partial t}. \tag{2.11}$$

It is convenient to introduce the usual velocity potential ϕ which satisfies the relations

$$u = \partial\phi/\partial x, \quad p = -\rho_0\,\partial\phi/\partial t,$$

and therefore the momentum equation is identically satisfied. We define the functions

$$W_S = \frac{\partial^2\phi}{\partial t^2} - a_S^2\frac{\partial^2\phi}{\partial x^2}, \quad W_T = \frac{\partial^2\phi}{\partial t^2} - a_T^2\frac{\partial^2\phi}{\partial x^2},$$

where the isentropic and isothermal speeds of sound in the undisturbed medium are given by

$$a_S^2 = \gamma p_0/\rho_0, \quad a_T^2 = p_0/\rho_0,$$

and γ is the ratio of specific heats, c_P/c_v.

By using the above definitions, the left-hand side of equation (2.10) becomes

$$\rho_0 c_P \frac{\partial T}{\partial t} - \frac{\partial p}{\partial t} = -\frac{\rho_0}{\gamma-1}\left(\frac{\partial^2\phi}{\partial t^2} - a_S^2\frac{\partial^2\phi}{\partial x^2}\right) = -\frac{\rho_0}{\gamma-1}W_S. \tag{2.12}$$

Similarly there results from equation (2.11),

$$\frac{\partial T}{\partial t} = -\frac{1}{R}\left(\frac{\partial^2\phi}{\partial t^2} - a_T^2\frac{\partial^2\phi}{\partial x^2}\right) = -\frac{W_T}{R}. \tag{2.13}$$

The right-hand side of equation (2.10) may be simplified by the following process. Differentiate equation (2.6) with respect to t and then linearize to obtain

$$\frac{\partial q_r}{\partial t} = -\frac{8\alpha\sigma T_0^3}{R}\left\{\int_0^x W_T E_2[\alpha(x-\bar{x})]\,d\bar{x} - \int_x^\infty W_T E_2[\alpha(\bar{x}-x)]\,d\bar{x} - \frac{R}{\alpha}\frac{dT_W}{dt}E_3(\alpha x)\right\}. \tag{2.14}$$

At this point it is convenient to substitute an approximate kernel of the form $ae^{-b\alpha x}$ for the correct kernel $E_2(\alpha x)$, and also $ae^{-b\alpha x}/b$ for $E_3(\alpha x)$. The constants a and b are determined by requiring that the area and first moment of the exponential kernel be equal to those of the exponential-integral kernel. It is found that $a = \frac{3}{4}$ and $b = \frac{3}{2}$. From the above equation, one obtains

$$\frac{\partial q_r}{\partial t} = -\frac{8\alpha\sigma T_0^3}{R}\left\{\int_0^x W_T \exp\{-b_1(x-\bar{x})\}\,d\bar{x} - \int_x^\infty W_T \exp\{-b_1(\bar{x}-x)\}\,d\bar{x} - \frac{R}{b_1}\frac{dT_W}{dt}e^{-b_1 x}\right\}, \tag{2.15}$$

where $b_1 = b\alpha$. By substitution of equations (2.12) and (2.15) into (2.10), there results

$$\frac{\partial W_S}{\partial t} = \frac{\gamma-1}{\rho_0}\frac{\partial^2 q_r}{\partial x\,\partial t} = -\frac{a_1^2}{2}\left\{-b_1\int_0^\infty W_T \exp(-b_1|x-\bar{x}|)\,d\bar{x} + 2W_T + R\frac{dT_W}{dt}e^{-b_1 x}\right\}, \tag{2.16}$$

where $$a_1^2 = 16a(\gamma-1)\,\alpha\sigma T_0^3/\rho_0 R.$$

Differential equation for the velocity potential

If equation (2.16) is differentiated twice, one obtains

$$\frac{\partial^3 W_S}{\partial x^2\,\partial t} = -\frac{a_1^2 b_1^2}{2}\left\{-b_1\int_0^\infty W_T \exp\{-b_1(x-\bar{x})\}\,d\bar{x} + 2W_T + R\frac{dT_W}{dt}e^{-b_1 x}\right\} - a_1^2\frac{\partial^2 W_T}{\partial x^2}. \tag{2.17}$$

By the substitution of equation (2.16) into (2.17) to eliminate the integral term, one obtains

$$\frac{\partial^3 W_S}{\partial x^2\,\partial t} + a_1^2\frac{\partial^2 W_T}{\partial x^2} - b_1^2\frac{\partial W_S}{\partial t} = 0, \tag{2.18}$$

or $$\frac{\partial^3}{\partial t\,\partial x^2}\left(\frac{\partial^2\phi}{\partial t^2} - a_S^2\frac{\partial^2\phi}{\partial x^2}\right) + a_1^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\phi}{\partial t^2} - a_T^2\frac{\partial^2\phi}{\partial x^2}\right) - b_1^2\frac{\partial}{\partial t}\left(\frac{\partial^2\phi}{\partial t^2} - a_S^2\frac{\partial^2\phi}{\partial x^2}\right) = 0. \tag{2.19}$$

The problem is now completely determined by the above equation with the following boundary conditions:

$$\begin{aligned}
t = 0:&\quad \phi = \phi_t = \phi_{tt} = 0;\\
x = 0:&\quad T_W = 0 \quad (t < 0),\\
&\qquad\;\; = \delta/R \quad (t > 0),\\
&\quad u = 0 \quad (t < 0),\\
&\qquad = \beta \quad (t > 0);\\
x \to \infty:&\quad \frac{\partial\phi}{\partial x} \to 0,\quad \frac{\partial\phi}{\partial t} \to 0.
\end{aligned}$$

Note that since heat conduction and viscosity have been neglected, the temperature of the gas at $x = 0$ is not necessarily equal to the temperature of the wall.

3. Solution of differential equation

Solution by Laplace transformation

If a Laplace transformation is applied to equation (2.19), the following differential equation is obtained

$$\delta_1 \frac{\partial^4 \bar{\phi}}{dx^4} - \delta_2 \frac{d^2 \bar{\phi}}{dx^2} + b_1^2 p^3 \bar{\phi} = 0, \tag{3.1}$$

where $\bar{\phi}$ is the Laplace transform of ϕ,

$$\bar{\phi} = \int_0^\infty e^{-pt} \phi \, dt,$$

and δ_1 and δ_2 are defined as

$$\delta_1 = a_S^2(p + a_1^2/\gamma), \quad \delta_2 = p(p^2 + a_1^2 p + b_1^2 a_S^2).$$

The solution of equation (3.1) can be found in the form

$$\bar{\phi} = A_1 e^{\gamma_1 x} + A_2 e^{\gamma_2 x}, \tag{3.2}$$

where
$$\gamma_{1,2} = -\left[\frac{\delta_2}{2\delta_1} \pm \frac{1}{2\delta_1}(\delta_2^2 - 4b_1^2 p^3 \delta_1)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

By the use of the inversion integral, the solution of equation (2.19) can then be written as

$$2\pi i \phi = \int_\Gamma A_1 \exp(pt + \gamma_1 x) \, dp + \int_\Gamma A_2 \exp(pt + \gamma_2 x) \, dp. \tag{3.3}$$

Γ is the path such that $\operatorname{Re} p = \text{const.}$ and to the right of all singularities.

The integration constants A_1 and A_2 are determined by requiring the solution (3.2) to satisfy the Laplace transform of the modified integro-differential equation (2.16). The result is

$$A_2 = \frac{\beta C_1 - \delta \gamma_1 p}{p(\gamma_2 C_1 - \gamma_1 C_2)}, \tag{3.4}$$

$$A_1 = \frac{\beta}{\gamma_1 p} - A_2 \frac{\gamma_2}{\gamma_1}, \tag{3.5}$$

where
$$C_1 = b_1(p^2 - a_T^2 \gamma_1^2)/(\gamma_1 + b_1), \tag{3.6}$$

$$C_2 = b_1(p^2 - a_T^2 \gamma_2^2)/(\gamma_2 + b_1). \tag{3.7}$$

Alternatively the solution (3.3) could be obtained directly from (2.16) by Laplace transform methods without forming the differential equation (2.19). However, the insight into the character of the solutions obtained from the knowledge and use of Whitham's investigations would be lost.

Since A_1, A_2, γ_1 and γ_2 are such complicated functions of p, a complete investigation of the solution would involve prohibitive labour. Simplifying approximations will be used in the following sections in the evaluation of the above integrals. To restrict further the amount of algebra required, the assumption will be made that $a_1^2/b_1 a_S \gg 1$. Since
$$a_1^2/b_1 a_S \sim \alpha \sigma T_0^4/\rho_0 C_v T_0,$$

this assumption is equivalent to restricting the investigation to the important and most interesting case when the temperature of the gas is very high and the radiative energy is much greater than the internal energy of the gas.

If $a_1^2/b_1 a_S \gg 1$, the approximate locations of the branch points of γ_2 are

$$\left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} = \frac{b_1^2 a_S^2}{a_1^2}\left(\frac{2-\gamma}{\gamma} \pm \frac{2i(\gamma-1)^{\frac{1}{2}}}{\gamma}\right), \tag{3.8}$$

$$\left.\begin{matrix} p_3 \\ p_4 \end{matrix}\right\} = -a_1^2 \pm 2b_1 a_S\left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{2}}; \tag{3.9}$$

γ_1 has the branch points p_1, p_2, p_3, p_4 and

$$p_5 = 0, \quad p_6 = -a_1^2/\gamma. \tag{3.10}$$

Approximation for small time

An approximate evaluation of the integrals occurring in equation (3.3) can be accomplished by substituting expansions for large p for the functions $A_{1,2}$ and $\gamma_{1,2}$. Since large p corresponds to high frequencies, this approximation is valid when the high-frequency waves dominate, i.e. when t is small or near discontinuities in the wave form.

For large p,

$$\gamma_1 = -\frac{p}{a_S} - \frac{a_1^2}{a_S}\left(\frac{\gamma-1}{2\gamma}\right) + O\left(\frac{1}{p}\right), \tag{3.11}$$

$$\gamma_2 = -b_1 + \frac{b_1 a_1^2}{2p} + O\left(\frac{1}{p^2}\right). \tag{3.12}$$

A_1 and A_2 can be approximated to the same order by carrying out the integrations in (3.3) and differentiating to find u and p, one obtains

$$\left.\begin{aligned} u &= \beta \exp\left(-\frac{\gamma-1}{2\gamma}a_1^2\frac{x}{a_S}\right) + \delta\tfrac{1}{4}a_1^2 b_1 t^2 \exp(-b_1 x) - \delta\tfrac{1}{4}a_1^2 b_1\left(t-\frac{x}{a_S}\right)^2 \\ &\qquad\qquad \times \exp\left(-\frac{\gamma-1}{2\gamma}a_1^2\frac{x}{a_S}\right) \quad (t > x/a_S), \\ &= \delta\tfrac{1}{4}a_1^2 b_1 t^2 e^{-b_1 x} \qquad\qquad (t < x/a_S), \end{aligned}\right\} \tag{3.13}$$

$$\left.\begin{aligned} p &= \beta\rho_0 a_S \exp\left(-\frac{\gamma-1}{2\gamma}a_1^2 t\right) + \delta\rho_0\tfrac{1}{2}a_1^2 t \exp(-b_1 x) - \delta\rho_0\tfrac{1}{4}a_1^2 b_1 a_S\left(t-\frac{x}{a_S}\right)^2 \\ &\qquad\qquad \times \exp\left(-\frac{\gamma-1}{2\gamma}a_1^2\frac{x}{a_S}\right) \quad (t > x/a_S), \\ &= \delta\tfrac{1}{2}\rho_0 a_1^2 t \exp(-b_1 x) \qquad\qquad (t < x/a_S). \end{aligned}\right\} \tag{3.14}$$

Asymptotic approximation for the isentropic wave

It can be shown in general that the effects of wall temperature are of lower order than the effects due to the wall motion. This can be seen for the particular case of small time from equations (3.13) and (3.14). In the following, the wall temperature effects will be neglected ($\delta = 0$) since the essential features of the wave motion can be found without this additional complication.

For large time, the form of the integrals in (3.3) suggests evaluation by the method of steepest descent. In this approximation, the dominant contributions to the integral come from the neighbourhood of the saddle-point and perhaps from any singularities enclosed by the contour path deformed to pass through the saddle-point. It will be seen that ϕ_1, the first integral in (3.3), asymptotically describes a wave which has the properties of an isothermal wave, while ϕ_2, the second integral in (3.3), asymptotically describes a wave which has the properties of an isentropic wave.

For the evaluation of ϕ_2, we anticipate that the saddle-point will be located near the origin. We then can approximate γ_2 and A_2 by expanding these functions for small p. It is found that

$$\gamma_2 = -\frac{p}{a_S} + \frac{\gamma-1}{2\gamma}\frac{a_1^2}{b_1^2 a_S^2}\frac{p^2}{a_S} + O(p^3), \tag{3.15}$$

$$A_2 = -\beta a_S/p^2 + O(1/p). \tag{3.16}$$

Write the second integral in (3.3) as $\int A_2 \exp\{f(p)\,t\}\,dp$, where

$$\begin{aligned} f(p) &= p + \gamma_2(x/t) \\ &= p\left(1-\frac{x}{a_S t}\right) + p^2\left(\frac{\gamma-1}{2\gamma}\right)\frac{a_1^2}{b_1^2 a_S^2}\frac{x}{a_S t}. \end{aligned} \tag{3.17}$$

Then the saddle-point is located at the point p_1 at which $f'(p_1) = 0$. From this, we find that

$$p_1 = \frac{x-a_S t}{x}\left(\frac{\gamma}{\gamma-1}\right)\frac{b_1^2 a_S^2}{a_1^2}. \tag{3.18}$$

The contribution to ϕ_2 along the path of steepest descent is then

$$\phi_2 = -\frac{\beta a_S}{2\pi}\exp\{f(p_1)\,t\}\int_{-\infty}^{\infty}\frac{\exp\{-\frac{1}{2}tf''(p_1)\,\tau^2\}}{(p_1+i\tau)^2}\,d\tau, \tag{3.19}$$

since the path of steepest descent is $\operatorname{Re} p = \text{const}$. The usual steepest-descent procedure has been modified by retaining $p^2 = (p_1+i\tau)^2$ in the integrand in order that the solution be valid near the wave front at $x = a_S t$, i.e. near $p_1 = 0$.

It can be shown that

$$\int_{-\infty}^{+\infty}\frac{\exp(-a^2\tau^2)}{(p_1+i\tau)^2}\,d\tau = -2\pi a^2|p_1|\exp(a^2p_1^2)\operatorname{erfc}(a|p_1|) + 2\pi^{\frac{1}{2}}a. \tag{3.20}$$

(3.19) then becomes

$$\phi_2 = +\beta a_S\left\{\frac{|x-a_S t|}{2a_S}\operatorname{erfc}\left(\frac{C_1|x-a_S t|}{x^{\frac{1}{2}}}\right) - C_2 x^{\frac{1}{2}}\exp\left(-\frac{C_1^2(x-a_S t)^2}{x}\right)\right\}, \tag{3.21}$$

where $$C_1 = \left(\frac{\gamma}{2(\gamma-1)\,a_S}\right)^{\frac{1}{2}}\frac{b_1 a_S}{a_1}, \quad C_2 = \left(\frac{\gamma-1}{2\pi\gamma a_S}\right)^{\frac{1}{2}}\frac{a_1}{b_1 a_S}.$$

The contribution from the singularity at the origin is

$$\left.\begin{aligned} \phi_2 &= -\beta a_S\left(t-\frac{x}{a_S}\right) \quad &(t > x/a_S), \\ &= 0 \quad &(t < x/a_S). \end{aligned}\right\} \tag{3.22}$$

If the contour path is taken as in figure 1, the dominant contributions are from the path of steepest descent near the real axis and from the singularity at $p = 0$ and other contributions are negligible in comparison.

Asymptotic approximation for the isothermal wave

In order to evaluate ϕ_1, the first integral of (3.3), by the method of steepest descent, we anticipate that the major portion of the wave front will be described when the saddle-point position is such that $b_1^2 a_S^2/a_1^2 < |p_1| < a_1^2/\gamma$. This assumption can always be checked once the solution is obtained. By expanding γ_1 and A_1 in this region, one finds that

$$\gamma_1 = -\frac{p}{a_T} + \frac{\gamma-1}{2}\frac{p^2}{a_1^2 a_T} - \frac{\gamma-1}{2\gamma}\frac{b_1^2 a_S^2}{a_1^2 a_T}, \tag{3.23}$$

$$A_1 = -\beta a_T/p^2. \tag{3.24}$$

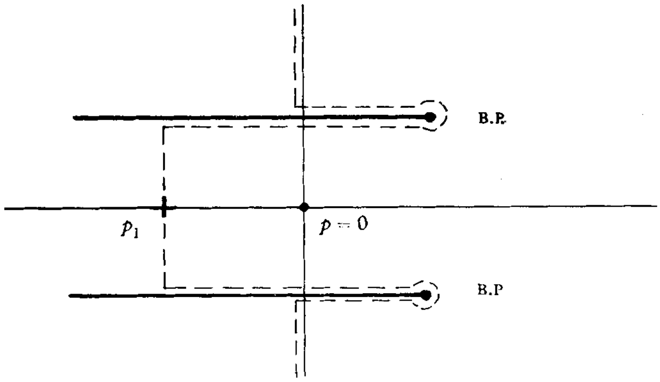

FIGURE 1. Contour path for the evaluation of ϕ_2, the isentropic wave.

If we write the first integral in (3.3) as $\int A_1 e^{f(p)t} dp$, then

$$f(p) = p\left(1 - \frac{x}{a_T t}\right) + \frac{\gamma-1}{2}\frac{x}{a_T t}\frac{p^2}{a_1^2} - \frac{\gamma-1}{2\gamma}\frac{b_1^2 a_S^2}{a_1^2}\frac{x}{a_T t}. \tag{3.25}$$

The position of the saddle-point is therefore

$$p_1 = \frac{x - a_T t}{x}\frac{a_1^2}{\gamma - 1}. \tag{3.26}$$

The contribution from the path of steepest descent is

$$\phi_1 = \beta a_T \exp\left(-\frac{\gamma-1}{2\gamma}\frac{b_1^2 a_S^2}{a_1^2}\frac{x}{a_T}\right)\left\{\frac{|x - a_T t|}{2a_T}\operatorname{erfc}\left(\frac{C_3|x - a_T t|}{x^{\frac{1}{2}}}\right) - C_4 x^{\frac{1}{2}} \exp\left(-C_3^2\frac{(x-a_T t)^2}{x}\right)\right\} \tag{3.27}$$

where $$C_3 = \left(\frac{a_1^2}{2(\gamma-1)\,a_T}\right)^{\frac{1}{2}}, \quad C_4 = \left(\frac{\gamma-1}{2\pi a_1^2 a_T}\right)^{\frac{1}{2}}.$$

To first order, the contribution from the singularities near the origin is

$$\left.\begin{aligned} \phi_1 &= -\beta a_T\left(t - \frac{x}{a_T}\right)\exp\left(-\frac{\gamma-1}{2\gamma}\frac{b_1^2 a_S^2}{a_1^2}\frac{x}{a_T}\right) && (t > x/a_T), \\ &= 0 && (t < x/a_T). \end{aligned}\right\} \tag{3.28}$$

Equations (3.27) and (3.28) describe the dominant contributions to ϕ_1.

4. Discussion

The nature of the solutions found in the previous section can be determined readily by examining a characteristic property of the waves, say the velocity of the gas, for the three limiting cases. The velocity is given by

I. *Small time*

$$\left.\begin{aligned} \frac{u}{\beta} &= \exp\left(-\frac{\gamma-1}{2\gamma} a_1^2 \frac{x}{a_S}\right) \quad && (t > x/a_S), \\ &= 0 && (t < x/a_S). \end{aligned}\right\} \tag{4.1}$$

II. *Isothermal wave*

$$\left.\begin{aligned} \frac{u}{\beta} &= \left\{1 - \tfrac{1}{2}\operatorname{erfc}\left(\frac{C_3|x-a_T t|}{x^{\frac{1}{2}}}\right) - \frac{C_4 a_T}{2x^{\frac{1}{2}}}\exp\left(-\frac{C_3^2(x-a_T t)^2}{x}\right)\right\} \\ &\qquad\qquad \times \exp\left(-\frac{\gamma-1}{2\gamma}\frac{b_1^2 a_S^2}{a_1^2}\frac{x}{a_T}\right) \quad (t > x/a_T), \\ &= \left\{\tfrac{1}{2}\operatorname{erfc}\left(\frac{C_3|x-a_T t|}{x^{\frac{1}{2}}}\right) - \frac{C_4 a_T}{2x^{\frac{1}{2}}}\exp\left(-\frac{C_3^2(x-a_T)^2 t}{x}\right)\right\} \\ &\qquad\qquad \times \exp\left(-\frac{\gamma-1}{2\gamma}\frac{b_1^2 a_S^2}{a_1^2}\frac{x}{a_T}\right) \quad (t < x/a_T). \end{aligned}\right\} \tag{4.2}$$

III. *Isentropic wave*

$$\left.\begin{aligned} \frac{u}{\beta} &= 1 - \tfrac{1}{2}\operatorname{erfc}\left(\frac{C_1|x-a_S t|}{x^{\frac{1}{2}}}\right) - \frac{C_2 a_S}{2x^{\frac{1}{2}}}\exp\left(-\frac{C_1^2(x-a_S t)^2}{x}\right) \quad (t > x/a_S), \\ &= \tfrac{1}{2}\operatorname{erfc}\left(\frac{C_1|x-a_S t|}{x^{\frac{1}{2}}}\right) - \frac{C_2 a_S}{2x^{\frac{1}{2}}}\exp\left(-\frac{C_1^2(x-a_S t)^2}{x}\right) \quad (t < x/a_S). \end{aligned}\right\} \tag{4.3}$$

From equation (4.1), it can be seen that, for small time, the wave front propagates at the isentropic speed of sound and decays exponentially with a characteristic decay length defined by

$$\frac{\gamma-1}{2\gamma}\frac{a_1^2 x}{a_S} = 1 \quad \text{or} \quad x = \frac{2\gamma}{\gamma-1}\frac{a_S}{a_1^2}. \tag{4.4}$$

That is, as the gas temperature becomes higher and therefore the radiative energy transfer increases, the wave attenuates more rapidly. The velocity u is discontinuous across the wave front.

The wave described by (4.2) propagates at the isothermal speed, eventually decays exponentially with a decay length

$$x = \frac{2\gamma}{\gamma-1}\frac{a_1^2}{b_1^2 a_S^2} a_T \tag{4.5}$$

and also diffuses. A characteristic diffusion length is defined by

$$\frac{a_1^2}{2(\gamma-1)}\frac{(x-a_T t)^2}{a_T x} = 1 \quad \text{or} \quad x - a_T t = \left[\frac{2(\gamma-1)}{a_1^2} a_T x\right]^{\frac{1}{2}}. \tag{4.6}$$

For some earlier time, the last term within the brackets in (4.2) is important and describes the growth of the wave.

The wave described by (4.3) propagates at the isentropic speed, grows since the second term becomes negligible as $x^{-\frac{1}{2}}$, diffuses with a diffusion width given by

$$x - a_S t = \left[\frac{2(\gamma-1)}{\gamma}\frac{a_1^2}{b_1^2 a_S^2} a_S x\right]^{\frac{1}{2}}, \tag{4.7}$$

but does not decay.

The general result is that as wave I decays, wave II begins to grow and reaches a maximum. As wave II decays and diffuses, wave III begins to grow and diffuse. A schematic diagram of this motion is shown in figure 2. The motion is as predicted earlier by analogy with Whitham's results.

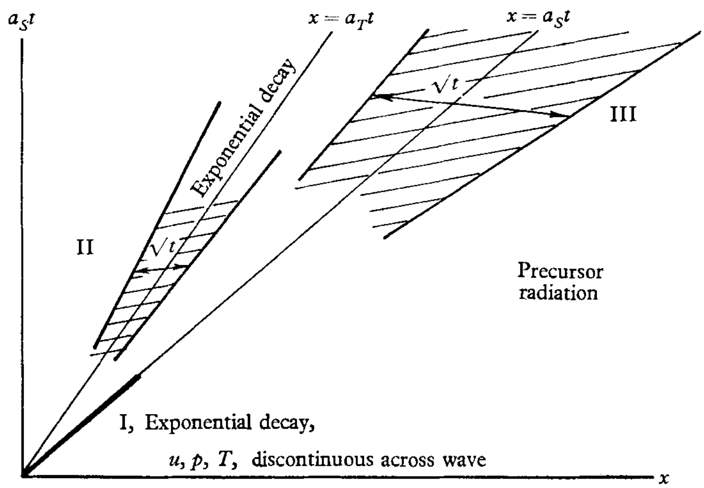

FIGURE 2. Schematic diagram of wave motion. I decays exponentially with u, p and T discontinuous across the wave. II diffuses with diffusion width proportional $\sqrt{t}$ and decays exponentially. III diffuses with a diffusion width proportional to $\sqrt{t}$. u, p and T are continuous across II and III. Disturbances are present in front of the waves due to precursor radiation.

It is of interest to investigate the diffusion zones of wave II and III more carefully. After a certain time when wave II has decayed appreciably, say when

$$x = \frac{2\gamma}{\gamma-1}\frac{a_1^2}{b_1^2 a_S^2} a_T,$$

then the diffusion width of II is given by

$$x - a_T t = 2/b_1 = O(1/\alpha). \tag{4.8}$$

But $1/\alpha$ corresponds to the mean free path for radiation. Therefore the width of the diffusion zone for the isothermal wave is less than a mean free path for radiation until it has lost most of its energy and has decayed. At the same time, the ratio of the diffusion widths of the waves II and III is given by

$$\frac{(x - a_T t)_{\mathrm{II}}}{(x - a_S T)_{\mathrm{III}}} = O\left(\frac{b_1 a_S}{a_1^2}\right) \ll 1. \tag{4.9}$$

The diffusion width of the isentropic wave is therefore much greater than the diffusion width of the isothermal wave and, in particular, much greater than a mean free path for radiation.

Because of its narrow diffusion zone, the isothermal wave loses energy and eventually disappears. The diffusion zone of the isentropic wave is comparatively wide. The radiative energy diffuses through the wave but is not lost from the wave. Hence this wave is essentially isentropic.

From equation (2.11) and the asymptotic solutions, the ratio of the temperature gradients at the wave fronts for waves II and III at a particular time can be determined and is given by

$$\frac{(\partial T/\partial t)_{\text{II}}}{(\partial T/\partial t)_{\text{III}}} = O\left(\frac{b_1 a_S}{a_1^2}\right) \ll 1. \tag{4.10}$$

Since the total change in temperature through a wave is proportional to the temperature gradient and wave width, it follows from (4.9) and (4.10) that

$$\frac{(\Delta T)_{\text{II}}}{(\Delta T)_{\text{III}}} = O\left(\frac{b_1^2 a_S^2}{a_1^4}\right) \ll 1. \tag{4.11}$$

The temperature change in the isothermal wave is hence much less than the temperature change in the isentropic wave and the wave is essentially isothermal.

There is a question as to what approximation is introduced by the exponential kernel substitution; in particular, whether the kernel substitution and differentiation technique has affected the main features of the wave motion. This can be checked most readily by applying a Fourier transform to equation (2.16). The following equation results,

$$\frac{d^3\tilde{\phi}}{dt^3} + C^2\left[-\tilde{E}_1(k) + \frac{3}{2\alpha}\right]\frac{d^2\tilde{\phi}}{dt^2} + a_S^2 k^2 \frac{d\tilde{\phi}}{dt} + C^2 a_T^2 k^2\left[-\tilde{E}_1(k) + \frac{3}{2\alpha}\right]\tilde{\phi} = 0, \tag{4.12}$$

where $\tilde{\phi}$ and $\tilde{E}_1'$ are the Fourier transforms of ϕ, E_1', i.e.

$$\tilde{\phi} = \int_{-\infty}^{+\infty} \phi e^{ikx} dx, \text{ etc.}$$

C is a constant, and $E_1' = ab\,e^{-b_1 x}$. If the exponential integral kernel is used, the resulting differential equation is identical except that $[-\hat{E}_1'(k) + 3/2\alpha]$ is replaced by $[-\tilde{E}_1(k) + 2/\alpha]$.

As $k \to \infty$, both $\tilde{E}_1$ and $\tilde{E}_1'$ disappear and the above brackets differ by a factor of 4/3. As $k \to 0$, the first three terms of both brackets are identical. Since only the first few terms are needed to evaluate the inversion integrals in the first approximation, it seems that to this approximation the exponential kernel substitution will not modify the essential results either for large or small time.

This research was sponsored by the National Science Foundation under Contract G-9445.

REFERENCES

KOURGANOFF, V. 1952 *Basic Methods in Transfer Problems*. Oxford University Press.

KROOK, M. 1955 On the solution of equations of transfer I. *Astrophys. J.* **122**, 488.

LICK, W. 1963 Energy transfer by radiation and conduction. *Proc. Heat Transf. Fluid Mech. Inst.* p. 14.

VINCENTI, W. & BALDWIN, B. 1962 Effect of thermal radiation on the propagation of plane acoustic waves. *J. Fluid Mech.* **12**, 449.

WHITHAM, G. B. 1959 Some comments on wave propagation and shock wave structure with application to magnetohydrodynamics. *Comm. Pure. Appl. Math.* **12**, 113.